Parallel Multigrid Equation Solver for Unstructured FEM Meshes

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Talk Outline

- Multigrid
- Prometheus
- PETSc
- T3E

Motivation

- Motivation: Large Scale Implicit Finite Element Method (FEM) Problems on Unstructured Meshes
- Problem: Solve sparse A in $Ax = b$ for x
- Direct Methods (LU factorization): $\sim O(n^2)$ for FEM matrices
- Iterative methods: Potentially $O(n)$ in time and space.
- Solution: Multilevel methods $\Rightarrow$ **multigrid**

Multigrid Basics

http://HTTP.CS.Berkeley.EDU/ demmel/cs267/lecture25/lecture25.html

Smoothers - Simple Matrix Splitting

$$Ax = b \Rightarrow (M - K)x = b$$

$$Mx = Kx + b$$

$$x = M^{-1}Kx + M^{-1}b$$

$$\hat{x}_{k+1} \leftarrow M^{-1}K\hat{x}_k + M^{-1}b$$

$$\hat{x}_{k+1} \leftarrow Smooth\,(A, (b - A\hat{x}_k))$$

Multiple Grids - Multiple "Scales of Resolution"

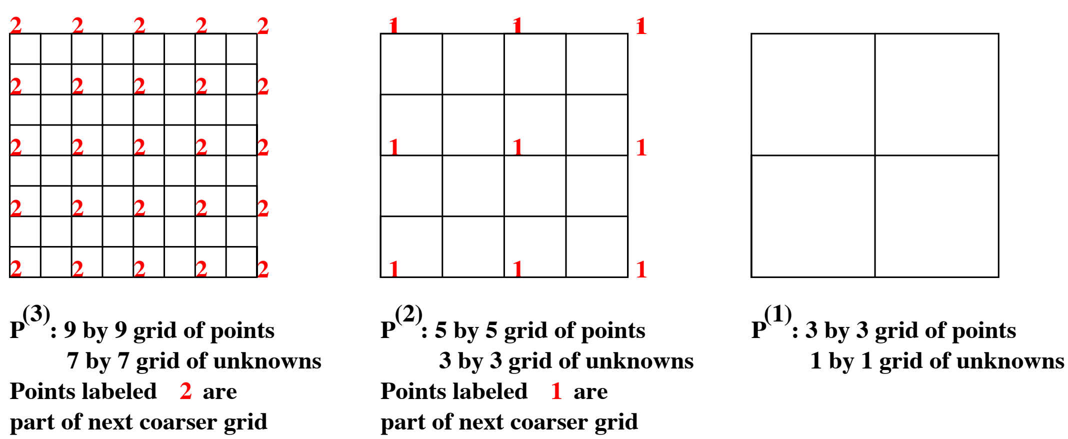


Figure 1: Multigrid coarse vertex set selection on structured meshes

Restriction R and Interpolation I Operators

$$\hat{x}_{k+1} \leftarrow R \cdot \hat{x}_k$$

$$e.g. \quad R(i,:) = [0 \quad 0 \quad 0 \quad 0 \quad \ldots \quad 1/2 \quad 1 \quad 1/2 \quad \ldots]$$

$$I = R^T$$

Galerkin Coarse Grid Operators

$$A_{i+1} = RA_iR^T$$

Multigrid Algorithm

$x = MultiGrid(A, b)$
- $if\ A.IsTop()$
  - $return\ A^{-1} \cdot b$
- $else$
  - $\hat{x} \leftarrow Smooth(A, b)$
  - $\hat{r} \leftarrow b - A \cdot \hat{x}$
  - $d \leftarrow MultiGrid(RAR^T, R \cdot \hat{r})$
  - $\hat{x} \leftarrow \hat{x} + R^T \cdot d$
  - $\hat{r} \leftarrow b - A \cdot \hat{x}$
  - $\hat{d} \leftarrow Smooth(A, \hat{r})$
  - $return\ \hat{x} + \hat{d}$
- $endif$

end

Algebraic MG

- Algebraic Architecture - Input Fine mesh. $A_{i+1} = RA_iR^T$.
- Algebraic Coarsening - make strongly connected cliques.
- Algebraic Interpolation operators (R^T)- minimize energy of coarse grids, and maintain compact support.

Prometheus - Multigrid Solver for Unstructured Grids

- Motivation: Large Scale Implicit Finite Element Problems on Unstructured Meshes
- Classical Multigrid (Geometric), with Algebraic Architecture
  - Guillard, 1992
  - Chan and Smith, 1994
- Evenly coarsen grid - Maximal Independent Sets
- Geometric remeshing of node set - Delaunay tessellation
- Finite element (FE) shape functions for interpolation operators
- Coarse grid matrices formed algebraically - Galerkin MG
  $$A_{i+1} = RA_iR^T$$
- http://www.mcs.anl.gov/CCST/research/discipline/index.html

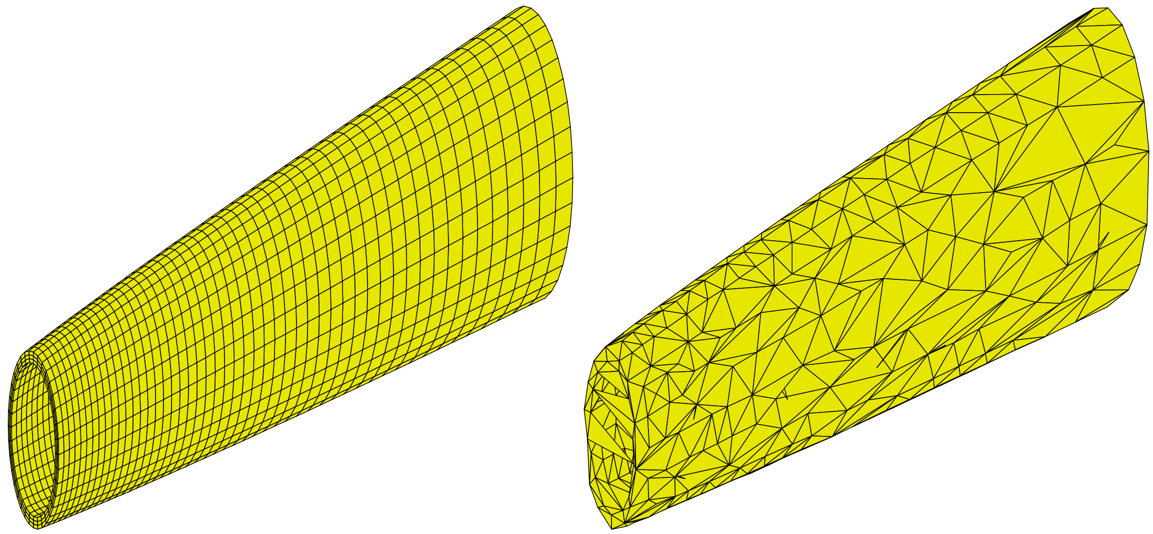


Figure 2: Sample Input Grid

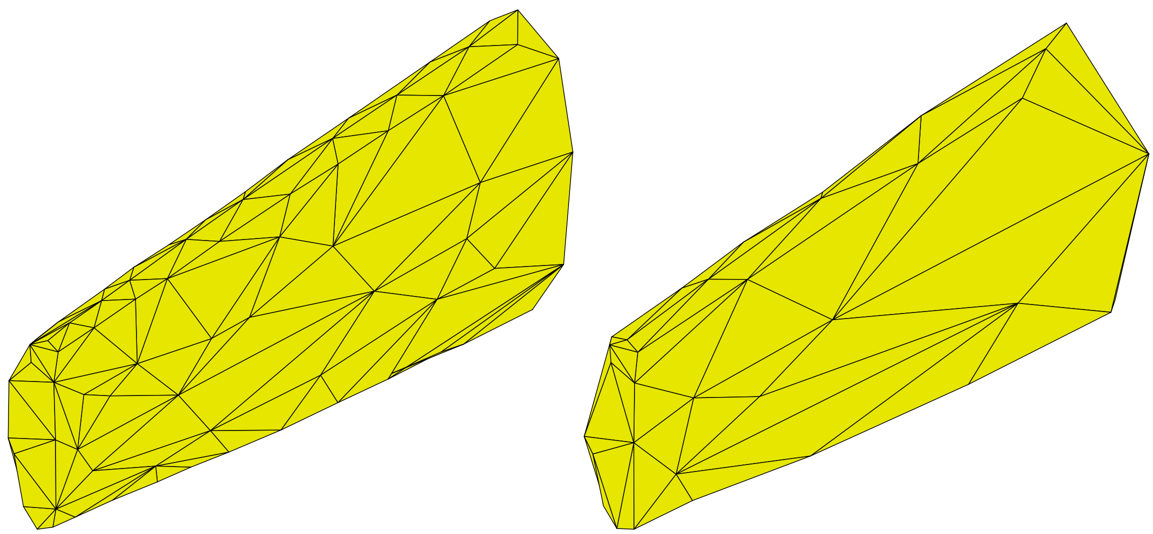
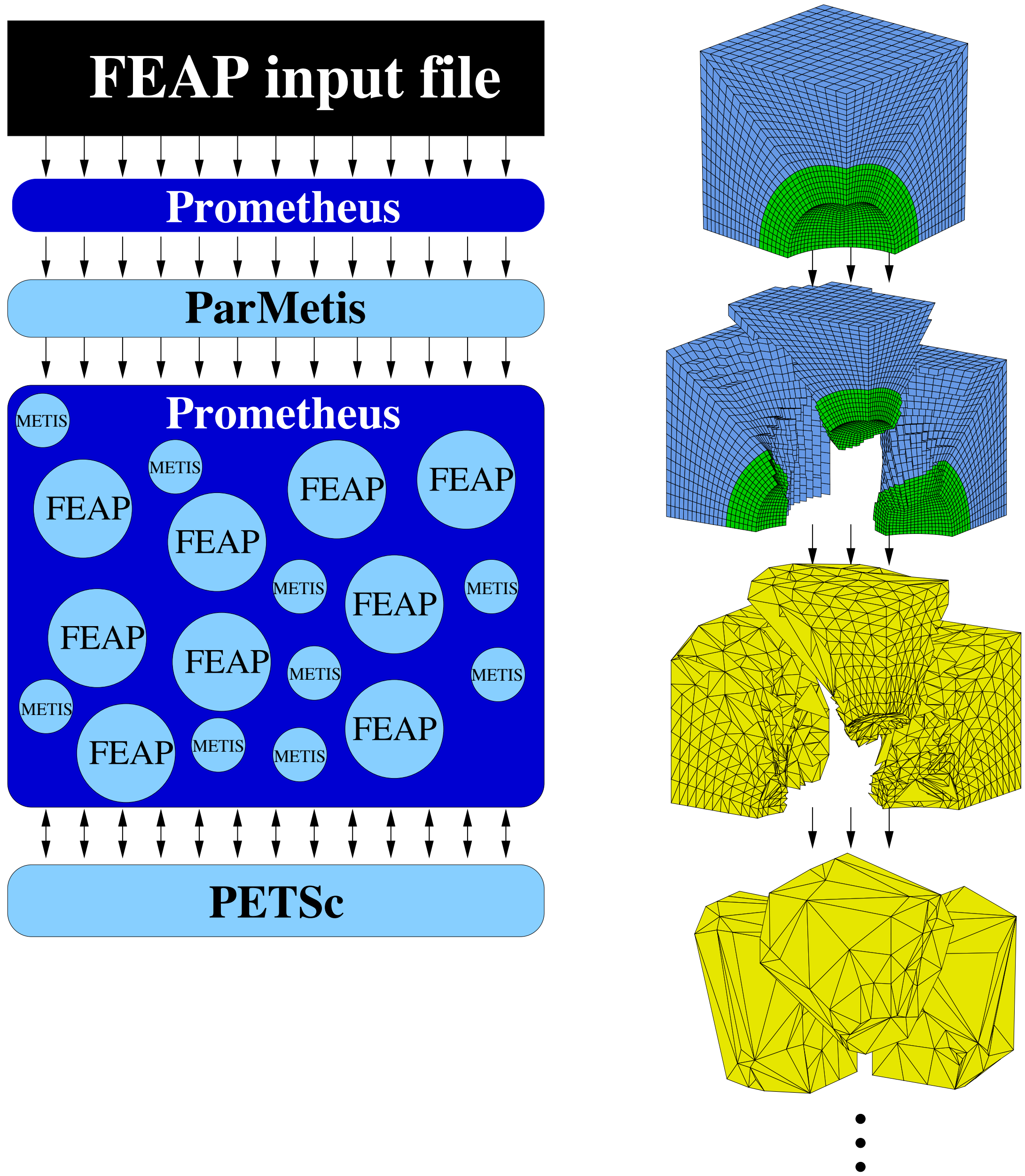
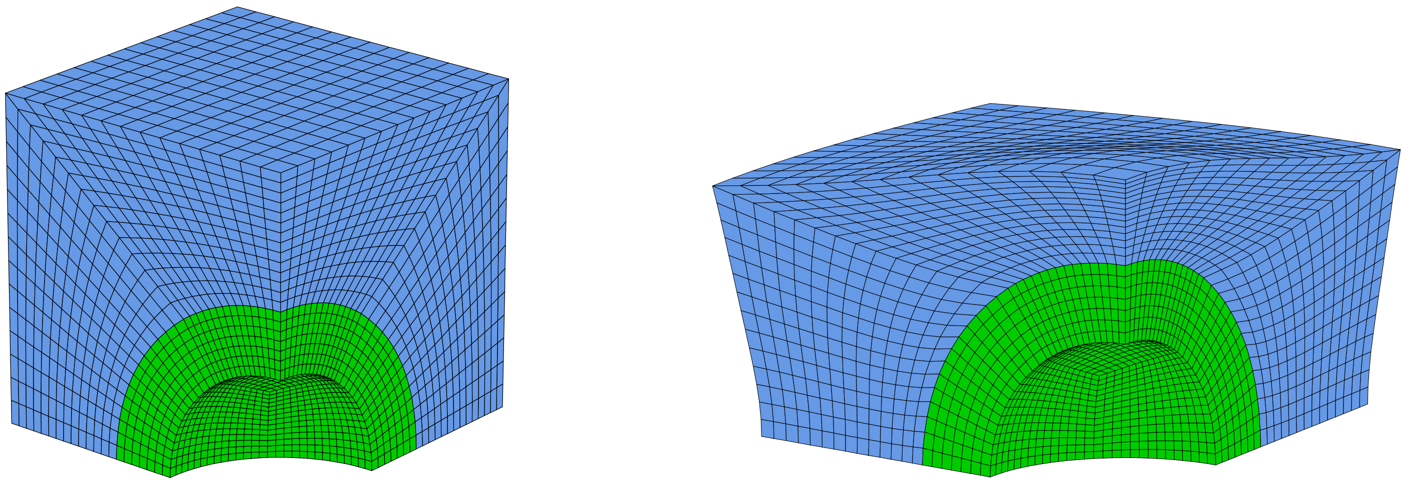


Figure 3: Sample Coarse Grids

Prometheus - parallel MG solver for FE matrices



Performance Results - Problem



13882 Vertex 3D FE mesh - Deformed Shape

- Parameterized mesh - 15,000 to 3,940,000 dof problems used
- Hard sphere covered by soft ($E_s = 10^{-4} E_h$) material
- Poisson ratio .49 for soft material
- About 15,000 dof per processor
- Linear elasticity

Prometheus Performance Results

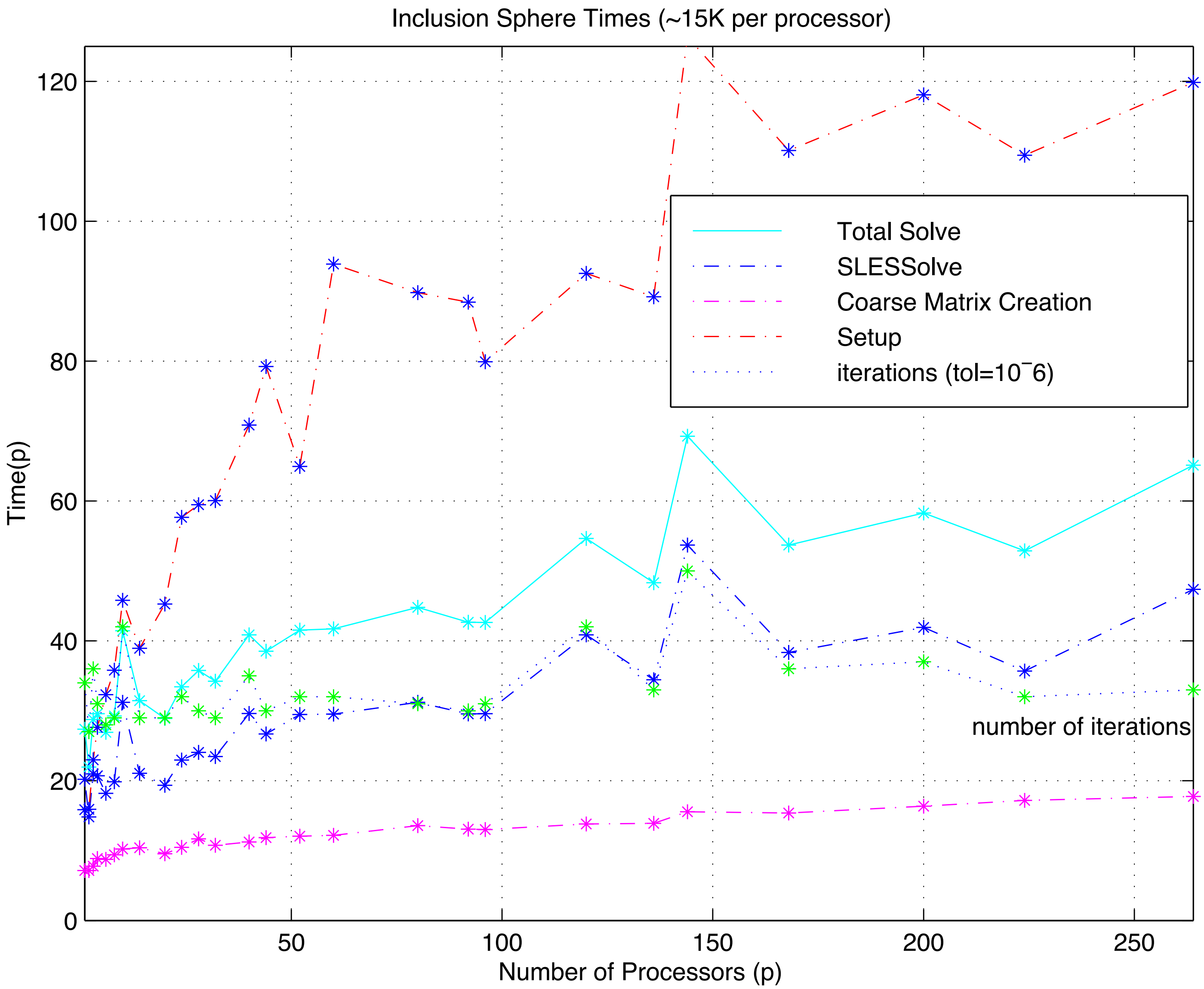


Figure 4: Parameterized Included Sphere Problem - Cray T3E

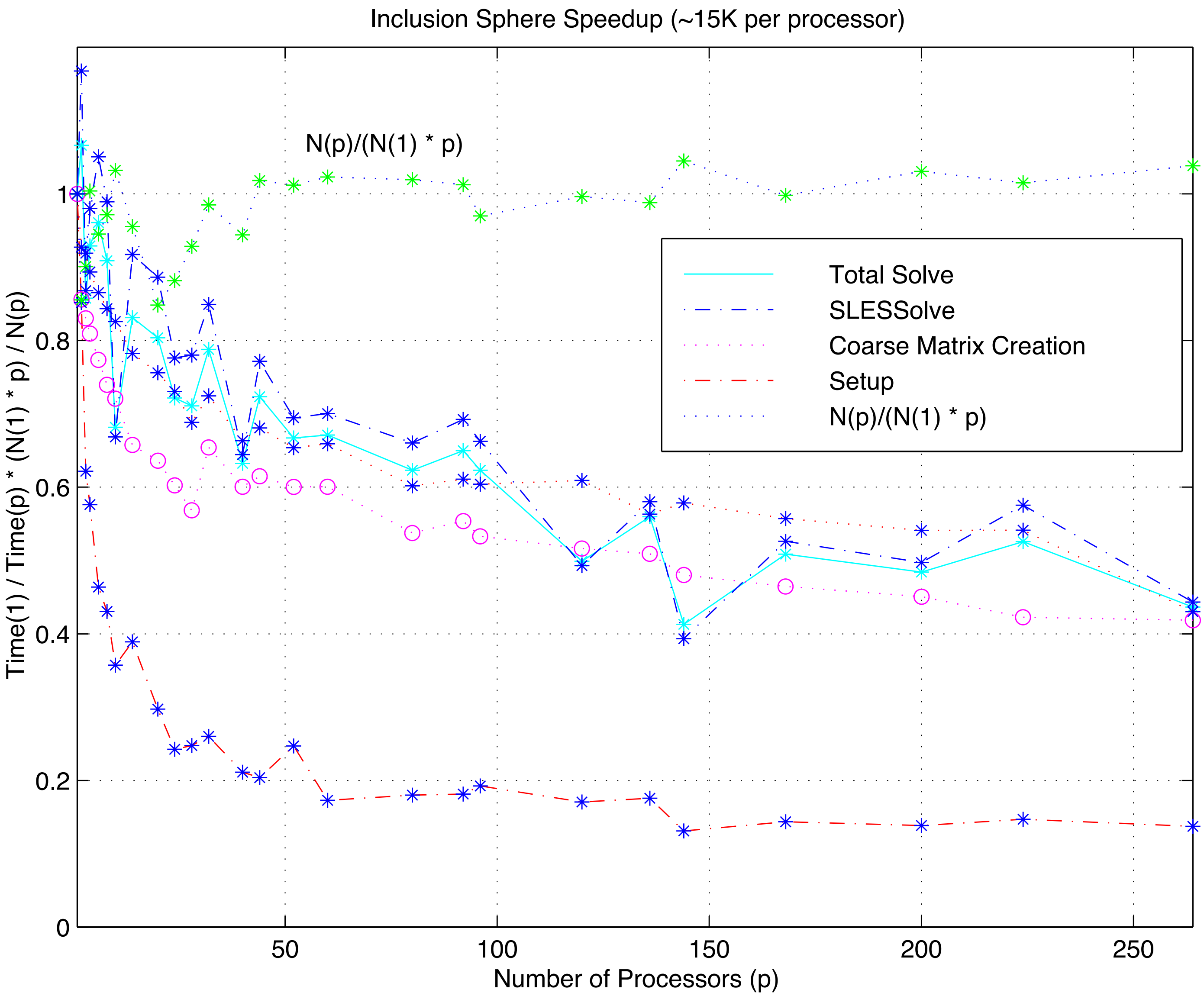


Figure 5: Parameterized Included Sphere Problem - Cray T3E

PETSc the Portable, Extensible Toolkit for Scientific Computation

- Numerical Libraries.
- Parallel Development Support.
- Object Oriented Library Design - implemented in ANSI C
- http://www.mcs.anl.gov/petsc