Parallel Multigrid Equation Solver for Unstructured FEM Meshes

April 16, 1998

Mark Adams (madams@cs.berkeley.edu)
Dept. of Civil Engineering, U. C. Berkeley
Supported by DOE

Research Advisors:

• R.L. Taylor (Civil Engineering)
• James Demmel (Computer Science/Math)
Talk Outline

- Multigrid
- Prometheus
- PETSc
- T3E
Motivation

- Motivation: Large Scale Implicit Finite Element Method (FEM) Problems on Unstructured Meshes
- Problem: Solve sparse $A$ in $Ax = b$ for $x$
- Direct Methods (LU factorization): $\sim O(n^2)$ for FEM matrices
- Iterative methods: Potentially $O(n)$ in time and space.
- Solution: Multilevel methods $\Rightarrow$ multigrid
Multigrid Basics


Smoothers - Simple Matrix Splitting

\[ Ax = b \Rightarrow (M - K)x = b \]
\[ Mx = Kx + b \]
\[ x = M^{-1}Kx + M^{-1}b \]
\[ \hat{x}_{k+1} \leftarrow M^{-1}K\hat{x}_k + M^{-1}b \]
\[ \hat{x}_{k+1} \leftarrow \text{Smooth}(A, (b - A\hat{x}_k)) \]

Multiple Grids - Multiple "Scales of Resolution"

\[ P^{(3)}: 9 \text{ by } 9 \text{ grid of points} \]
\[ 7 \text{ by } 7 \text{ grid of unknowns} \]
Points labeled 2 are part of next coarser grid

\[ P^{(2)}: 5 \text{ by } 5 \text{ grid of points} \]
\[ 3 \text{ by } 3 \text{ grid of unknowns} \]
Points labeled 1 are part of next coarser grid

\[ P^{(1)}: 3 \text{ by } 3 \text{ grid of points} \]
\[ 1 \text{ by } 1 \text{ grid of unknowns} \]

Figure 1: Multigrid coarse vertex set selection on structured meshes
Restriction $R$ and Interpolation $I$ Operators

\[ \hat{x}_{k+1} \leftarrow R \cdot \hat{x}_k \]

e.g. \[ R(i,:) = [0 \ 0 \ 0 \ 0 \ \cdots \ 1/2 \ 1 \ 1/2 \ \cdots ] \]

\[ I = R^T \]

Galerkin Coarse Grid Operators

\[ A_{i+1} = RA_i R^T \]

Multigrid Algorithm

\[ x = \text{MultiGrid}(A, b) \]

if $A.$IsTop()

\[ \text{return } A^{-1} \cdot b \]

else

\[ \hat{x} \leftarrow \text{Smooth}(A, b) \]

\[ \hat{r} \leftarrow b - A \cdot \hat{x} \]

\[ d \leftarrow \text{MultiGrid}(RAR^T, R \cdot \hat{r}) \]

\[ \hat{x} \leftarrow \hat{x} + R^T \cdot d \]

\[ \hat{r} \leftarrow b - A \cdot \hat{x} \]

\[ \hat{d} \leftarrow \text{Smooth}(A, \hat{r}) \]

\[ \text{return } \hat{x} + \hat{d} \]

endif

end
Algebraic MG

- Algebraic Architecture - Input Fine mesh. $A_{i+1} = RA_iR^T$.
- Algebraic Coarsening - make strongly connected cliques.
- Algebraic Interpolation operators ($R^T$) - minimize energy of coarse grids, and maintain compact support.
Prometheus - Multigrid Solver for Unstructured Grids

- Motivation: Large Scale Implicit Finite Element Problems on Unstructured Meshes

- Classical Multigrid (Geometric), with Algebraic Architecture
  
  - Guillard, 1992
  
  - Chan and Smith, 1994

- Evenly coarsen grid - Maximal Independent Sets

- Geometric remeshing of node set - Delaunay tessellation

- Finite element (FE) shape functions for interpolation operators

- Coarse grid matrices formed algebraically - Galerkin MG

\[ A_{i+1} = RA_iR^T \]

Figure 2: Sample Input Grid

Figure 3: Sample Coarse Grids
Prometheus - parallel MG solver for FE matrices
Performance Results - Problem

13882 Vertex 3D FE mesh - Deformed Shape

- Parameterized mesh - 15,000 to 3,940,000 dof problems used
- Hard sphere covered by soft \((E_s = 10^{-4} E_h)\) material
- Poisson ratio .49 for soft material
- About 15,000 dof per processor
- Linear elasticity
Prometheus Performance Results

![Graph showing performance results for different numbers of processors. The graph includes lines for Total Solve, SLESSolve, Coarse Matrix Creation, Setup, and iterations (tol=10^-6). The x-axis represents the number of processors, and the y-axis represents time. The graph shows that as the number of processors increases, the time for each process generally decreases.](image)

Figure 4: Parameterized Included Sphere Problem - Cray T3E
Inclusion Sphere Speedup (~15K per processor)

Figure 5: Parameterized Included Sphere Problem - Cray T3E
PETSc the Portable, Extensible Toolkit for Scientific Computation

• Numerical Libraries.
• Parallel Development Support.
• Object Oriented Library Design - implemented in ANSI C
• http://www.mcs.anl.gov/petsc